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二维准晶平面问题中的 Hamilton 体系求解方法*

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摘要： 针对二维准晶平面问题,该文通过导入 Hamilton 体系,将问题转化为 Hamilton 体系下的辛本征值和辛本征解问题,即问题的解可由辛本征解组成的级数表示.利用辛本征解之间的辛共轭正交关系,可将满足边界条件的解问题归结为代数方程组的求解问题,从而形成一种解析求解方法.这种方法可直接推广到求解混合边界条件及分段边界条件问题中.

关　键　词： 二维准晶；　Hamilton 体系；　辛共轭正交；　辛方法；　平面问题

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A Hamiltonian System Solution Method for Planar Problems of 2D Quasicrystals

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Abstract: Aimed at the planar problem of 2D quasicrystals, the problem was transformed into one of symplectic eigenvalues and symplectic eigensolutions through introduction of the Hamiltonian system. In the Hamiltonian system, the solution to this problem was expressed by a series of symplectic eigensolutions. With the symplectic conjugate orthogonality relationship between symplectic eigensolutions, the solving problem satisfying boundary conditions can be reduced to a problem of solving algebraic equations, thus to form an analytical solution method. The proposed method can be directly extended to solve the problems of mixed boundary conditions and segmented boundary conditions.

Key words: 2D quasicrystal; Hamiltonian system; symplectic conjugate orthogonality; symplectic method; planar problem

0　引　　言

准晶因其重量轻、强度高和性能稳定[1-2]以及优异的耐磨性、耐腐蚀性、隔热性等[3]特性被广泛关注,成为凝聚态物理和材料科学研究的热门课题.由于准晶自身特殊的微观结构,在国防军工、新能源等领域表现出广阔的应用前景,如太阳光能吸收[4]、热电换能[5]和储氢结构设计[6]等.

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准晶材料与晶体材料有所不同,其具有多场的特征[7],即声子场和相位子场,且对一个或多个方向表现出不同于晶体周期性的平动和旋转对称性,或称之为准周期方向.为了描述准晶材料的准周期对称性引入高维空间(四维、五维或六维)[8],这个高维空间又分为两个子空间:一个是物理空间,又称为平行空间,描述准晶体中的声子场特性;另一个是互补空间,又称垂直空间,描述准晶体中的相位子场特性.根据准晶的准周期方向可将其分为一维、二维和三维准晶等[9].事实上,这些特征体现在准晶材料的本构关系[10]上.在准晶材料表征的基础上,人们开始对准晶材料和结构的力学行为展开研究分析.其研究主要集中在准晶基本解[11]、准晶通解[12]、准晶热弹性平面问题[13]、准晶接触问题[14]、准晶半空间问题[15]、准晶断裂问题[16]、准晶导波问题[17]和准晶梁板壳诸问题[18-21].人们采用各种方法对这些问题展开研究,然而缺乏一般性的方法.在这些关于准晶的研究中,经常需要对高阶的偏微分方程进行求解,面对这些复杂的微分方程,目前更多的是采用数值或者是半解析方法进行求解,只有少数非常特殊的条件下才可以得到问题的解析通解.因此对于更多的复杂情况下的微分方程,寻求一种完全理性的和直接的解析求解方法,对于更加深刻地理解准晶结构的各种力学特性和行为具有重要的意义.辛方法是一种直接的解析求解方法,并被 Zhong(钟万勰)[22]首次引入到应用弹性力学中.这种方法与传统的半逆解法或逆解法不同,即通过引入对偶变量,将原问题转换到 Hamilton 体系中,并利用辛本征解所构成的完备解空间,对问题解析求解.文献[23-25]针对准晶平面问题,采用逆方法提出特殊情况下的辛系统,并分析了等效合力问题[26].因而,构造 Hamilton 体系,并建立直接的通用方法是一个迫切的问题.

本文采用对能量变分得到以原变量和对偶变量(应力)表示具有守恒性的 Hamilton 正则方程组.在 Hamilton 体系下,将边界条件分成两部分,即侧边条件和端部条件.利用侧边条件求解辛本征值本征解,并借助于辛本征解之间的辛共轭正交关系得到问题解的级数表达式,再将级数解满足端部条件归结为确定级数系数的代数方程组的问题,从而形成一种求解准晶相关问题的直接方法和一般性方法.该方法可推广到求解混合边界条件及分段边界条件问题[27]等.

1 基本问题

考虑二维十次对称准晶矩形区域平面应力问题[9],并选取直角坐标系 xOy 且两个方向均为准周期方向.记声子场和相位子场位移向量为 $\boldsymbol{u}=\{u_x,u_y\}^{\mathrm{T}}$ 和 $\boldsymbol{w}=\{w_x,w_y\}^{\mathrm{T}}$.几何方程和本构方程[9-10]分别为

$$\varepsilon_x=\partial_x u_x,\ \varepsilon_y=\partial_y u_y,\ \varepsilon_{xy}=\frac{1}{2}(\partial_y u_x+\partial_x u_y),\ \omega_x=\partial_x w_x,\ \omega_y=\partial_y w_y,\ \omega_{xy}=\partial_y w_x,\ \omega_{yx}=\partial_x w_y,\tag{1}$$

$$\begin{cases}\boldsymbol{\sigma}=\boldsymbol{C}\boldsymbol{\varepsilon}+\boldsymbol{R}\boldsymbol{\omega},\\ \boldsymbol{H}=\boldsymbol{R}^{\mathrm{T}}\boldsymbol{\varepsilon}+\boldsymbol{K}\boldsymbol{\omega},\end{cases}\tag{2}$$

其中,声子场应力 $\boldsymbol{\sigma}=\{\sigma_x,\sigma_y,\sigma_{xy}\}^{\mathrm{T}}$,声子场应变 $\boldsymbol{\varepsilon}=\{\varepsilon_x,\varepsilon_y,\varepsilon_{xy}\}^{\mathrm{T}}$,相位子场应力 $\boldsymbol{H}=\{H_x,H_y,H_{xy},H_{yx}\}^{\mathrm{T}}$,相位子场应变 $\boldsymbol{\omega}=\{\omega_x,\omega_y,\omega_{xy},\omega_{yx}\}^{\mathrm{T}}$;用 Lamé 常数 λ 和 μ 表示的声子场弹性常数矩阵 $\boldsymbol{C}$、声子场和相位子场耦合系数矩阵 $\boldsymbol{R}$ 以及相位子场弹性常数矩阵 $\boldsymbol{K}$ 如下:

$$\boldsymbol{C}=\begin{bmatrix}\lambda+2\mu & \lambda & \\ \lambda & \lambda+2\mu & \\ & & 2\mu\end{bmatrix},\ \boldsymbol{R}=\begin{bmatrix}R & R & & \\ -R & -R & & \\ & & -R & R\end{bmatrix},\ \boldsymbol{K}=\begin{bmatrix}K_1 & K_2 & & \\ K_2 & K_1 & & \\ & & K_1 & -K_2\\ & & -K_2 & K_1\end{bmatrix}.$$

用广义位移表示的基本方程[9]可写成

$$\begin{cases}(\lambda+2\mu)\partial_x^2u_x+\mu\partial_y^2u_x+(\lambda+\mu)\partial_{xy}^2u_y+R\partial_x^2w_x-R\partial_y^2w_x+2R\partial_{xy}^2w_y=0,\\ \mu\partial_x^2u_y+(\lambda+2\mu)\partial_y^2u_y+(\lambda+\mu)\partial_{xy}^2u_x-2R\partial_{xy}^2w_x+R\partial_x^2w_y-R\partial_y^2w_y=0,\\ R\partial_x^2u_x-R\partial_y^2u_x-2R\partial_{xy}^2u_y+K_1\partial_x^2w_x+K_1\partial_y^2w_x=0,\\ R\partial_x^2u_y-R\partial_y^2u_y+2R\partial_{xy}^2u_x+K_1\partial_y^2w_y+K_1\partial_x^2w_y=0.\end{cases}\tag{3}$$

侧边条件[27-28]可表述为

$$\begin{cases} y = h: \sigma_y = 0,\ \sigma_{xy} = 0,\ H_y = 0,\ H_{yx} = 0, \\ y = -h: \sigma_y = 0,\ \sigma_{xy} = 0,\ H_y = 0,\ H_{yx} = 0. \end{cases} \tag{4}$$

这里应该指出,对于非齐次方程(3)和侧边条件(4),可通过一个特解将其转化为齐次问题.

2　Hamilton 体系的构造

以 x 方向模拟广义时间,并记为 $\dot{f} \equiv \partial f/\partial x$.广义应变能表达式,即 Lagrange 函数可表示为

$$\begin{aligned} L = [&(\lambda + 2\mu)(\dot{u}_x)^2 + \mu(\partial_y u_x)^2 + (\lambda + 2\mu)(\partial_y u_y)^2 + \mu(\dot{u}_y)^2 + K_1(\dot{w}_x)^2 + K_1(\partial_y w_x)^2 + \\ &K_1(\partial_y w_y)^2 + K_1(\dot{w}_y)^2 + 2\lambda \dot{u}_x \partial_y u_y + 2\mu \dot{u}_y \partial_y u_x + 2R\dot{u}_x \dot{w}_x - 2R(\partial_y u_x)(\partial_y w_x) + \\ &2R\dot{u}_x \partial_y w_y + 2R\dot{w}_y \partial_y u_x - 2R\dot{w}_x \partial_y u_y - 2R\dot{u}_y \partial_y w_x + 2R\dot{u}_y \dot{w}_y - 2R(\partial_y u_y)(\partial_y w_y) + \\ &2K_2 \dot{w}_x \partial_y w_y - 2K_2 \dot{w}_y \partial_y w_x]/2. \end{aligned} \tag{5}$$

可以证明,对 Lagrange 函数(5)的变分,可以得到基本方程(3).以广义位移 u_x, u_y, w_x, w_y 为原基本变量,通过变分可以得到其对偶变量:

$$\begin{cases} p_1 = \dfrac{\delta L}{\delta \dot{u}_x} = (\lambda + 2\mu)\dot{u}_x + R\dot{w}_x + \lambda \partial_y u_y + R\partial_y w_y = \sigma_x, \\ p_2 = \dfrac{\delta L}{\delta \dot{u}_y} = \mu \dot{u}_y + R\dot{w}_y + \mu \partial_y u_x - R\partial_y w_x = \sigma_{xy}, \\ p_3 = \dfrac{\delta L}{\delta \dot{w}_x} = R\dot{u}_x + K_1 \dot{w}_x - R\partial_y u_y + K_2 \partial_y w_y = H_x, \\ p_4 = \dfrac{\delta L}{\delta \dot{w}_y} = R\dot{u}_y + K_1 \dot{w}_y + R\partial_y u_x - K_2 \partial_y w_x = H_{yx}. \end{cases} \tag{6}$$

可见,对偶变量的物理意义分别表示如下: σ_x 表示外法线方向为 x 平面上沿 x 方向的声子场正应力, σ_{xy} 表示外法线方向为 x 平面上沿 y 方向的声子场切应力, H_x 表示外法线方向为 x 平面上沿 x 方向的相位子场正应力, H_{yx} 表示外法线方向为 y 平面上沿 x 方向的相位子场切应力.记原变量向量为 $\boldsymbol{q}$, 对偶变量向量为 $\boldsymbol{p}$, 即

$$\begin{cases} \boldsymbol{q} = \{u_x, u_y, w_x, w_y\}^{\mathrm{T}} \equiv \{q_1, q_2, q_3, q_4\}^{\mathrm{T}}, \\ \boldsymbol{p} = \{\sigma_x, \sigma_{xy}, H_x, H_{yx}\}^{\mathrm{T}} \equiv \{p_1, p_2, p_3, p_4\}^{\mathrm{T}}, \end{cases} \tag{7}$$

Lagrange 函数(5)可用 $\boldsymbol{q}$ 和 $\boldsymbol{p}$ 表示为 $L(\boldsymbol{q}, \boldsymbol{p})$.进而,Hamilton 函数可以由 Lagrange 函数表示,即

$$H(\boldsymbol{q}, \boldsymbol{p}) = \dot{u}_x \sigma_x + \dot{u}_y \sigma_{xy} + \dot{w}_x H_x + \dot{w}_y H_{yx} - L(\boldsymbol{q}, \boldsymbol{p}). \tag{8}$$

由此可导出 Hamilton 正则方程组:

$$\begin{cases} \dot{\boldsymbol{q}} = \delta H/\delta \boldsymbol{p}, \\ \dot{\boldsymbol{p}} = -\delta H/\delta \boldsymbol{q}, \end{cases} \tag{9}$$

即

$$\begin{Bmatrix} \dot{\boldsymbol{q}} \\ \dot{\boldsymbol{p}} \end{Bmatrix} = \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \\ \boldsymbol{C} & -\boldsymbol{A}^{\mathrm{T}} \end{bmatrix} \begin{Bmatrix} \boldsymbol{q} \\ \boldsymbol{p} \end{Bmatrix}, \tag{10}$$

其中矩阵

$$\boldsymbol{A} = \begin{bmatrix} 0 & -\eta_2 \partial_y & 0 & -\eta_3 \partial_y \\ -\partial_y & 0 & \eta_4 \partial_y & 0 \\ 0 & \eta_5 \partial_y & 0 & -\eta_6 \partial_y \\ 0 & 0 & \eta_7 \partial_y & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} \eta_8 & 0 & -\eta_9 & 0 \\ 0 & \eta_{10} & 0 & -\eta_{11} \\ -\eta_9 & 0 & \eta_{12} & 0 \\ 0 & -\eta_{11} & 0 & \eta_{13} \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & -\eta_{14} \partial_y^2 & 0 & \eta_{15} \partial_y^2 \\ 0 & 0 & -\eta_{16} \partial_y^2 & 0 \\ 0 & \eta_{15} \partial_y^2 & 0 & -\eta_{17} \partial_y^2 \end{bmatrix}, \quad -\boldsymbol{A}^{\mathrm{T}} = \begin{bmatrix} 0 & -\partial_y & 0 & 0 \\ -\eta_2 \partial_y & 0 & \eta_5 \partial_y & 0 \\ 0 & \eta_4 \partial_y & 0 & \eta_7 \partial_y \\ -\eta_3 \partial_y & 0 & -\eta_6 \partial_y & 0 \end{bmatrix},$$

这里的系数定义为

$$\begin{cases}\eta_0=(\lambda+2\mu)K_1-R^2,\ \eta_1=\mu K_1-R^2,\ \eta_2=(R^2+\lambda K_1)/\eta_0,\\ \eta_3=R(K_1-K_2)/\eta_0,\ \eta_4=(K_1-K_2)R/\eta_1,\ \eta_5=2(\lambda+\mu)R/\eta_0,\\ \eta_6=[(\lambda+2\mu)K_2-R^2]/\eta_0,\ \eta_7=(\mu K_2-R^2)/\eta_1,\\ \eta_8=K_1/\eta_0,\ \eta_9=R/\eta_0,\ \eta_{10}=K_1/\eta_1,\ \eta_{11}=R/\eta_1,\\ \eta_{12}=(\lambda+2\mu)/\eta_0,\ \eta_{13}=\mu/\eta_1,\ \eta_{14}=4(\lambda+\mu)(\mu K_1-R^2)/\eta_0,\\ \eta_{15}=2(\lambda+\mu)(K_1-K_2)R/\eta_0,\ \eta_{16}=(K_1-K_2)(\mu K_1+\mu K_2-2R^2)/\eta_1,\\ \eta_{17}=(K_1-K_2)[(\lambda+2\mu)(K_2+K_1)-2R^2]/\eta_0.\end{cases}$$

定义全状态向量为 $\boldsymbol{\psi}=\{\boldsymbol{q}^{\mathrm{T}},\boldsymbol{p}^{\mathrm{T}}\}^{\mathrm{T}}$, Hamilton 正则方程(10)可以写成

$$\boldsymbol{\psi}=\boldsymbol{H}\boldsymbol{\psi}. \tag{11}$$

采用广义分离变量法,其解可以写成 $\boldsymbol{\psi}=\boldsymbol{\varphi}(y)\mathrm{e}^{\xi x}$, 因而,有

$$(\boldsymbol{H}-\xi\boldsymbol{I})\boldsymbol{\varphi}(y)=0, \tag{12}$$

这里 ξ 为本征值.在导出 Hamilton 正则方程组(10)的同时,可得到侧边条件:

$$\begin{cases}\eta_2p_1-\eta_5p_3+\eta_{14}\partial_yq_2-\eta_{15}\partial_yq_4=0, & p_2=0,\\ \eta_3p_1+\eta_6p_3-\eta_{15}\partial_yq_2+\eta_{17}\partial_yq_4=0, & p_4=0,\end{cases}\quad y=\pm h. \tag{13}$$

3 辛本征值和辛本征解

通过求解方程(12)且满足侧边条件(13),可得到 12 个零辛本征值 ($\xi=0$) 本征解及约当型本征解:

$$\begin{cases}\boldsymbol{\psi}_{01}^{(\alpha)}(x,y)=\{1,0,0,0,0,0,0,0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{02}^{(\alpha)}(x,y)=\{0,0,1,0,0,0,0,0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{03}^{(\alpha)}(x,y)=\{-y,x,0,0,0,0,0,0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{04}^{(\alpha)}(x,y)=\{\eta_4\eta_7^{-1}y,0,\eta_7^{-1}y,x,0,0,0,0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{05}^{(\alpha)}(x,y)=\{0,1,0,0,0,0,0,0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{06}^{(\alpha)}(x,y)=\{0,0,0,1,0,0,0,0\}^{\mathrm{T}},\end{cases} \tag{14}$$

$$\begin{cases}\boldsymbol{\psi}_{01}^{(\beta)}(x,y)=\{x,-\eta_2y,0,-\eta_5y,\eta_{14},0,\eta_{15},0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{02}^{(\beta)}(x,y)=\{0,\eta_4y,x,-\eta_6y,\eta_{15},0,\eta_{17},0\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{03}^{(\beta)}(x,y)=\left\{-xy,\frac{1}{2}(x^2+\eta_2y^2),0,\frac{1}{2}\eta_5y^2,-\eta_{14}y,0,-\eta_{15}y,0\right\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{04}^{(\beta)}(x,y)=\left\{\eta_4\eta_7^{-1}xy,\eta_{18}y^2,\eta_7^{-1}xy,\frac{1}{2}x^2-\eta_{19}y^2,\eta_{20}y,0,\eta_{21}y,0\right\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{05}^{(\beta)}(x,y)=\left\{-\eta_{22}y^3-\frac{1}{2}x^2y+\eta_{23}h^2y,\frac{1}{2}\eta_2xy^2+\frac{1}{6}x^3,\eta_{24}y^3+\eta_{25}h^2y,\right.\\ \qquad\left.\frac{1}{2}\eta_5xy^2,-\eta_{14}xy,\eta_{26}(y^2-h^2),-\eta_{15}xy,\eta_{27}(y^2-h^2)\right\}^{\mathrm{T}},\\ \boldsymbol{\psi}_{06}^{(\beta)}(x,y)=\left\{-\eta_{28}y^3+\frac{1}{2}\eta_4\eta_7^{-1}x^2y+\eta_{29}h^2y,\eta_{18}xy^2,\right.\\ \qquad-\eta_{30}y^3+\frac{1}{2}\eta_7^{-1}x^2y+\eta_{31}h^2y,-\eta_{19}xy^2+\frac{1}{6}x^3,\\ \qquad\left.\eta_{20}xy,-\eta_{32}(y^2-h^2),\eta_{21}xy,-\eta_{33}(y^2-h^2)\right\}^{\mathrm{T}},\end{cases} \tag{15}$$

其中参数

$$
\begin{cases}
\eta_{18}=(\eta_3-2\eta_2\eta_{15}\eta_{14}^{-1})\eta_7^{-1}/2,\ \eta_{19}=(\eta_{15}\eta_{11}+\eta_6)\eta_7^{-1}/2,\ \eta_{20}=3\eta_{15}\eta_7^{-1},\\
\eta_{21}=(\eta_{17}+\eta_4\eta_{15})\eta_7^{-1},\ \eta_{22}=(\eta_2-2\eta_5\eta_{10}\eta_{11}^{-1})/6,\\
\eta_{23}=[(\eta_7^{-1}-3)\eta_5\eta_{11}-\eta_{14}\eta_1^{-1}]\eta_{13}^{-1}/2,\ \eta_{24}=\eta_5/2,\\
\eta_{25}=\eta_5(\eta_7^{-1}-3)/2,\ \eta_{26}=\eta_{14}/2,\ \eta_{27}=3\eta_{15}/2,\ \eta_{28}=(\eta_3+\eta_{10}\eta_{15})\eta_7^{-1}/2,\\
\eta_{29}=3(2\eta_{10}-\eta_4\eta_{11}\eta_7^{-1})\eta_{15}\eta_7^{-1}/4,\ \eta_{30}=(\eta_6+4\eta_{11}\eta_{15}+\eta_4\eta_{13}\eta_{11}^{-1})\eta_7^{-1}/6,\\
\eta_{31}=3\eta_{11}\eta_{15}(3-\eta_7^{-1})\eta_7^{-1}/4,\ \eta_{32}=3\eta_{15}\eta_7^{-1}/2,\ \eta_{33}=9\eta_4\eta_{15}\eta_7^{-1}/4.
\end{cases}
$$

这里的零辛本征值本征解中 $\boldsymbol{\psi}_{01}^{(\alpha)}$,$\boldsymbol{\psi}_{02}^{(\alpha)}$ 和 $\boldsymbol{\psi}_{03}^{(\alpha)}$ 的物理意义为声子场的刚体平动和转动;而 $\boldsymbol{\psi}_{04}^{(\alpha)}$,$\boldsymbol{\psi}_{05}^{(\alpha)}$ 和 $\boldsymbol{\psi}_{06}^{(\alpha)}$ 为相位子场的刚体平动和转动;$\boldsymbol{\psi}_{01}^{(\beta)}$,$\boldsymbol{\psi}_{02}^{(\beta)}$ 和 $\boldsymbol{\psi}_{03}^{(\beta)}$ 分别为声子场简单拉伸、纯弯曲和剪力弯曲问题的解;$\boldsymbol{\psi}_{04}^{(\beta)}$,$\boldsymbol{\psi}_{05}^{(\beta)}$ 和 $\boldsymbol{\psi}_{06}^{(\beta)}$ 为相位子场简单拉伸、纯弯曲和剪力弯曲问题的解.同时,可求解出对称问题和反对称问题的非零辛本征值 ($\xi\neq 0$) 本征解,分别表示为

$$
\begin{cases}
q_1=-C_2\kappa_1\cos(\xi y)+C_3[-\kappa_2\cos(\xi y)+2\kappa_1(\xi y)\sin(\xi y)]+C_4[\kappa_{13}\cos(\xi y)-\\
\quad 3\kappa_2(\xi y)\sin(\xi y)-3\kappa_1(\xi y)^2\cos(\xi y)],\\
q_2=\kappa_1C_2\sin(\xi y)+C_3[-\kappa_5\sin(\xi y)+2\kappa_1(\xi y)\cos(\xi y)]+C_4[\kappa_4\sin(\xi y)+\\
\quad \kappa_3(\xi y)\cos(\xi y)+3\kappa_1(\xi y)^2\sin(\xi y)],\\
q_3=C_1\cos(\xi y)+C_2(\xi y)\sin(\xi y)+C_3(\xi y)^2\cos(\xi y)+C_4(\xi y)^3\sin(\xi y),\\
q_4=-C_1\sin(\xi y)+C_2(\xi y)\cos(\xi y)-C_3(\xi y)^2\sin(\xi y)+C_4[\kappa_6\sin(\xi y)+(\xi y)^3\cos(\xi y)],\\
p_1=-\kappa_7\xi C_2\cos(\xi y)+[-2\kappa_7\cos(\xi y)+2\kappa_7(\xi y)\sin(\xi y)]\xi C_3+\\
\quad [\kappa_8\cos(\xi y)-6\kappa_7(\xi y)\sin(\xi y)-3\kappa_7(\xi y)^2\cos(\xi y)]\xi C_4,\\
p_2=\kappa_7\xi C_2\sin(\xi y)+2\kappa_7(\xi y)\cos(\xi y)\xi C_3+[\kappa_9\sin(\xi y)+3\kappa_7(\xi y)^2\sin(\xi y)]\xi C_4,\\
p_3=\kappa_{10}\xi C_1\cos(\xi y)+[-\kappa_{10}\cos(\xi y)+\kappa_{10}(\xi y)\sin(\xi y)]\xi C_2+[2\kappa_{10}(\xi y)\sin(\xi y)+\\
\quad \kappa_{10}(\xi y)^2\cos(\xi y)]\xi C_3+[\kappa_{11}\cos(\xi y)-3\kappa_{10}(\xi y)^2\cos(\xi y)+\kappa_{10}(\xi y)^3\sin(\xi y)]\xi C_4,\\
p_4=-\kappa_{10}\xi C_1\sin(\xi y)+[\kappa_{10}\sin(\xi y)+\kappa_{10}(\xi y)\cos(\xi y)]\xi C_2+[2\kappa_{10}(\xi y)\cos(\xi y)-\\
\quad \kappa_{10}(\xi y)^2\sin(\xi y)]\xi C_3+[\kappa_{12}\sin(\xi y)+3\kappa_{10}(\xi y)^2\sin(\xi y)+\kappa_{10}(\xi y)^3\cos(\xi y)]\xi C_4,
\end{cases}
\tag{16}
$$

$$
\begin{cases}
q_1=C_6\kappa_1\sin(\xi y)+C_7[-\kappa_2\sin(\xi y)-2\kappa_1(\xi y)\cos(\xi y)]+C_8[-\kappa_{13}\sin(\xi y)-\\
\quad 3\kappa_2(\xi y)\cos(\xi y)+3\kappa_1(\xi y)^2\sin(\xi y)],\\
q_2=C_6\kappa_1\cos(\xi y)+C_7[\kappa_5\cos(\xi y)+2\kappa_1(\xi y)\sin(\xi y)]+C_8[\kappa_4\cos(\xi y)-\\
\quad \kappa_3(\xi y)\sin(\xi y)+3\kappa_1(\xi y)^2\cos(\xi y)],\\
q_3=C_5\sin(\xi y)+C_6(\xi y)\cos(\xi y)+C_7(\xi y)^2\sin(\xi y)+C_8(\xi y)^3\cos(\xi y),\\
q_4=C_5\cos(\xi y)-C_6(\xi y)\sin(\xi y)+C_7(\xi y)^2\cos(\xi y)+\\
\quad C_8[\kappa_6\cos(\xi y)-(\xi y)^3\sin(\xi y)],\\
p_1=\kappa_7\sin(\xi y)\xi C_6+[-2\kappa_7\sin(\xi y)-2\kappa_7(\xi y)\cos(\xi y)]\xi C_7+[-\kappa_8\sin(\xi y)-\\
\quad 6\kappa_7(\xi y)\cos(\xi y)+3\kappa_7(\xi y)^2\sin(\xi y)]\xi C_8,\\
p_2=\kappa_7\cos(\xi y)\xi C_6+2\kappa_7(\xi y)\sin(\xi y)\xi C_7+[\kappa_9\cos(\xi y)+3\kappa_7(\xi y)^2\cos(\xi y)]\xi C_8,\\
p_3=\kappa_{10}\sin(\xi y)\xi C_5+[\kappa_{10}\sin(\xi y)+\kappa_{10}(\xi y)\cos(\xi y)]\xi C_6+[-2\kappa_{10}(\xi y)\cos(\xi y)+\\
\quad \kappa_{10}(\xi y)^2\sin(\xi y)]\xi C_7+[-\kappa_{11}\sin(\xi y)+3\kappa_{10}(\xi y)^2\sin(\xi y)+\kappa_{10}(\xi y)^3\cos(\xi y)]\xi C_8,\\
p_4=\kappa_{10}\cos(\xi y)\xi C_5+[\kappa_{10}\cos(\xi y)-\kappa_{10}(\xi y)\sin(\xi y)]\xi C_6+[2\kappa_{10}(\xi y)\sin(\xi y)+\\
\quad \kappa_{10}(\xi y)^2\cos(\xi y)]\xi C_7+[\kappa_{12}\cos(\xi y)+3\kappa_{10}(\xi y)^2\cos(\xi y)-\kappa_{10}(\xi y)^3\sin(\xi y)]\xi C_8,
\end{cases}
\tag{17}
$$

其中,$C_k(k=1,2,\cdots,8)$ 为本征解待定系数及材料参数:

$$\begin{cases}\kappa_1 = K_1/(2R),\ \kappa_2 = \eta_1/((\lambda+\mu)R),\ \kappa_3 = 3\eta_0/((\lambda+\mu)R),\\ \kappa_4 = 3[(-\lambda+3\mu)K_1 - 4\eta_0]/[4(\lambda+\mu)R],\\ \kappa_5 = \eta_0/[(\lambda+\mu)R],\ \kappa_6 = 3[(\lambda+2\mu)\eta_1 + \mu\eta_0]/[2(\lambda+\mu)R^2],\ \kappa_7 = \eta_1/R,\\ \kappa_8 = 3[\lambda\mu K_1 - (\lambda+2\mu)\eta_1 - (\lambda+\mu)\eta_0 + \mu R^2 - K_1(\lambda^2+\lambda\mu+2\mu^2)]/[4(\lambda+\mu)R],\\ \kappa_9 = 3[\mu R^2 - \mu\eta_0 + 2(\lambda+2\mu)\eta_1 - \lambda\mu K_1]/[4(\lambda+\mu)R],\ \kappa_{10} = K_1 - K_2,\\ \kappa_{11} = 3[2((\lambda+2\mu)\mu K_1K_2 + R^4) - (\lambda+3\mu)(K_1+K_2)R^2]/[2(\lambda+\mu)R^2],\\ \kappa_{12} = 3[2\eta_1\eta_0 - \eta_0R^2 + 2(\lambda+2\mu)\eta_1K_1 + R^4 + \lambda K_1R^2]/[4(\lambda+\mu)R^2],\\ \kappa_{13} = -3(3\eta_1+\eta_0)/[4(\lambda+\mu)R].\end{cases}$$

非零辛本征值 ($\xi \neq 0$) 本征解满足侧边条件(13),可分别得到对应对称问题和反对称问题的代数方程组为

$$\boldsymbol{A}^{(1)}\boldsymbol{c}_1 = \boldsymbol{0},\ \boldsymbol{A}^{(2)}\boldsymbol{c}_2 = \boldsymbol{0}. \tag{18}$$

由方程组(18)的系数行列式为零,可得到非零辛本征值所满足的特征方程:

$$|\boldsymbol{A}^{(1)}| = 0,\ |\boldsymbol{A}^{(2)}| = 0, \tag{19}$$

其中 $\boldsymbol{c}_1 = \{C_1, C_2, C_3, C_4\}^{\mathrm{T}}$ 和 $\boldsymbol{c}_2 = \{C_5, C_6, C_7, C_8\}^{\mathrm{T}}$, 矩阵 $\boldsymbol{A}^{(1)}$ 和 $\boldsymbol{A}^{(2)}$ 的分量分别为

$$\begin{cases}A_{11}^{(k)} = 0,\ A_{12}^{(k)} = (-1)^{k-1}\eta_{11}^{-1}f^{(k)},\ A_{13}^{(k)} = 2\eta_{11}^{-1}[-f^{(k)} + (-1)^k\bar{\xi}g^{(k)}],\\ A_{14}^{(k)} = 3(-1)^{k-1}(\kappa_8 + \kappa_7\bar{\xi}^2)f^{(k)} - 6\eta_{11}^{-1}\bar{\xi}g^{(k)},\ A_{21}^{(k)} = -\kappa_{10}f^{(k)},\ A_{22}^{(k)} = -\kappa_{10}\bar{\xi}g^{(k)},\\ A_{23}^{(k)} = -\kappa_{10}\bar{\xi}^2f^{(k)},\ A_{24}^{(k)} = (-1)^{k-1}\kappa_{12}f^{(k)} - \kappa_{10}\bar{\xi}^3g^{(k)},\ A_{31}^{(k)} = 0,\ A_{32}^{(k)} = \eta_{11}^{-1}g^{(k)},\\ A_{33}^{(k)} = 2\eta_{11}^{-1}\bar{\xi}f^{(k)},\ A_{34}^{(k)} = -3(\kappa_{15} - \kappa_7\bar{\xi}^2)g^{(k)},\ A_{41}^{(k)} = (-1)^k\kappa_{10}g^{(k)},\\ A_{42}^{(k)} = \kappa_{10}(g^{(k)} + (-1)^{k-1}\bar{\xi}f^{(k)}),\ A_{43}^{(k)} = \kappa_{10}(2\bar{\xi}f^{(k)} + (-1)^k\bar{\xi}^2g^{(k)}),\\ A_{44}^{(k)} = 3(\kappa_{14} + \kappa_{10}\bar{\xi}^2)g^{(k)} + (-1)^{k-1}\kappa_{10}\bar{\xi}^3f^{(k)},\end{cases} \quad k = 1,2,$$

这里

$$\begin{cases}f^{(1)}(\bar{\xi}) = \cos(\bar{\xi}),\\ f^{(2)}(\bar{\xi}) = \sin(\bar{\xi}),\end{cases}\ \begin{cases}g^{(1)}(\bar{\xi}) = \sin(\bar{\xi}),\\ g^{(2)}(\bar{\xi}) = \cos(\bar{\xi}),\end{cases}\quad \bar{\xi} \equiv \xi h,$$

以及参数

$$\begin{cases}\kappa_{14} = (4K_1\mu\eta_0 - 8R^2\eta_1 - 4R^4 - 4K_1\lambda R^2)/(4R^2(\lambda+\mu)),\\ \kappa_{15} = (2\lambda R^2 + \mu R^2 - \mu\eta_0 - \lambda\mu K_1)/(4R(\lambda+\mu)),\\ \kappa_{16} = 3R(3\lambda^2 + 22\lambda\mu + 16\mu^2)/(4(\lambda+\mu)),\ \kappa_{17} = 3R(\lambda^2 - 4\lambda\mu + 3\mu^2)/(\lambda+\mu),\\ \kappa_{18} = -3R^2(6\lambda + 5\mu)/(4(\lambda+\mu)),\ \kappa_{19} = 3R^2,\\ \kappa_{20} = 3K_1(\lambda^3 + 7\lambda^2\mu + 20\lambda\mu^2 + 12\mu^3)/(4R(\lambda+\mu)),\\ \kappa_{21} = 3K_1\mu(\lambda^2 + 3\lambda\mu + 2\mu^2)/(R(\lambda+\mu)),\\ \kappa_{22} = \eta_8\kappa_{16} + \eta_9\kappa_{18} - \eta_8\kappa_{20},\ \kappa_{23} = \eta_9\kappa_{19} + \eta_8\kappa_{21} - \eta_8\kappa_{17}.\end{cases}$$

将方程(19)得到的非零辛本征值 $\bar{\xi} = \bar{\xi}_n (n = 1,2,3,\cdots)$, 对应代入代数方程组(18),可确定待定系数.我们注意到,此时的代数方程组是线性相关的,因而待定系数为非独立的.可以验证,每一组系数存在一个独立的常数,其他的系数可由该常数表示.不妨设该常数为 1,并将这些系数分别代入式(16)和式(17),就得到了对应非零辛本征值的辛本征解.

4 辛本征解之间的辛共轭正交关系和数值方法

通过求解特征方程(19),可得两组系列本征值 $\xi = \xi_n^{(1)}$ 和 $\xi = \xi_n^{(2)} (n = 1,2,\cdots)$.针对所得本征值,利用方

程(19)可建立系数之间的关系,并由式(16)和式(17)得到辛本征解表达式.不妨将两组本征值交替重新排列,并记为 $\xi=\xi_n(n=1,2,\cdots)$, 这样,就有与之对应的辛本征解.注意到,辛本征解成对出现,即 $\pm\xi_n(n=1,2,\cdots)$.因此,可将辛本征解记为 $\psi_n^{(\xi_n)}$ (当 $\mathrm{Re}\,\xi_n>0$, 或 $\mathrm{Re}\,\xi_n=0$ 和 $\mathrm{Im}\,\xi_n>0$) 和 $\psi_n^{(-\xi_n)}$ (当 $\mathrm{Re}\,\xi_n<0$, 或 $\mathrm{Re}\,\xi_n=0$ 和 $\mathrm{Im}\,\xi_n<0$), 引入泛函

$$\langle\psi_1,\psi_2\rangle=\int_{-h}^{h}(\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{p}_2-\boldsymbol{p}_1^{\mathrm{T}}\boldsymbol{q}_2)\,\mathrm{d}y \tag{20}$$

和辛归一因子 $\rho_n=1/\langle\psi_n^{(\xi_n)},\psi_n^{(-\xi_n)}\rangle$, 并分别定义 α 类和 β 类本征解 $\psi_n^{(\alpha)}=\psi_n^{(\xi_n)}$ 和 $\psi_n^{(\beta)}=\rho_n\psi_n^{(-\xi_n)}$.这样,辛本征解之间存在辛共轭正交关系:

$$\begin{cases}\langle\psi_n^{(\alpha)},\psi_m^{(\beta)}\rangle=\langle\psi_{0n}^{(\alpha)},\psi_{0m}^{(\beta)}\rangle=-\langle\psi_n^{(\beta)},\psi_m^{(\alpha)}\rangle=-\langle\psi_{0n}^{(\beta)},\psi_{0m}^{(\alpha)}\rangle=\delta_{nm},\\ \langle\psi_n^{(\alpha)},\psi_m^{(\alpha)}\rangle=\langle\psi_{0n}^{(\alpha)},\psi_{0m}^{(\alpha)}\rangle=-\langle\psi_n^{(\beta)},\psi_m^{(\beta)}\rangle=-\langle\psi_{0n}^{(\beta)},\psi_{0m}^{(\beta)}\rangle=0.\end{cases} \tag{21}$$

事实上,问题的解应该由零本征值本征解和非零本征值本征解组成,即

$$\psi(x,y)=\sum_{n=1}^{6}[c_n^{(0)}\psi_{0n}^{(\alpha)}(x,y)+d_n^{(0)}\psi_{0n}^{(\beta)}(x,y)]+\sum_{n=1}^{\infty}[a_n\psi_n^{(\alpha)}(y)\mathrm{e}^{\xi_n x}+b_n\psi_n^{(\beta)}(y)\mathrm{e}^{-\xi_n x}]. \tag{22}$$

上式中第一部分是由零本征值本征解表示的 Saint-Venant 问题的解,其中的 α 类和 β 类分别描述了单位位移和应力情况,其基本顺序是可以调整的.在讨论等效问题时只需采用该 Saint-Venant 解即可.式中第二部分是由非零本征值本征解表示的 Saint-Venant 原理所覆盖的解,在级数中大于 6 项以后皆为非零本征值本征解.在讨论边界效应问题时,则必须考虑该项.事实上,非零本征值本征解的 α 类和 β 类分别为刻画左端和右端的主项,即 α 类和 β 类分别主导了左端和右端的边界效应.从解(22)可以看出,利用 Hamilton 体系方法,可将问题的求解问题归结为确定级数的系数问题.为了数值方法的表示更加简单,不妨将式(22)中的零本征值本征解部分和非零本征值本征解部分重新排序,可将式(22)统一记为级数

$$\psi(x,y)=\sum_{n=1}^{\infty}[a_n\psi_n^{(\alpha)}(x,y)+b_n\psi_n^{(\beta)}(x,y)]. \tag{23}$$

考虑长度 $2l$ 和宽度 $2h$ 的平面矩形区域,侧边条件(13)及端部条件.注意到在端部条件中, H_{xy} 不属于广义对偶变量,但可由广义原变量和对偶变量表示,即 $H_{xy}=-\eta_4p_2-\eta_7p_4+\eta_{16}(\partial_y q_3)=\tilde{H}_{xy}$.端部条件包括广义位移条件、广义应力条件、混合端部条件和分段混合边界条件等.不妨以一种典型的边界条件[27-28]为例,即

$$\begin{cases}x=-l:\ q_1=0,\ q_2=0,\ q_3=0,\ q_4=0,\\ x=l:\ \sigma_x=\tilde{\sigma}_x,\ \sigma_{xy}=\tilde{\sigma}_{xy},\ H_x=\tilde{H}_x,\ p_4-\eta_{34}(\partial_y q_3)=-\eta_{36}\tilde{H}_{xy}-\eta_{35}\tilde{\sigma}_{xy},\end{cases} \tag{24}$$

其中, $\eta_{34}=\eta_{16}/\eta_7$, $\eta_{35}=\eta_4/\eta_7$ 和 $\eta_{36}=1/\eta_7$.为方便数值求解,取式(23)中本征解级数的前 N 项.根据辛共轭正交关系(20)和(21),并利用边界条件(24),可得确定解(23)系数的代数方程组:

$$\begin{cases}\sum_{n=1}^{N}[A_{nm}^{(1)}a_n+B_{nm}^{(1)}b_n]=0,\\ \sum_{n=1}^{N}[A_{nm}^{(2)}a_n+B_{nm}^{(2)}b_n]=C_m^{(2)},\end{cases}\qquad m=1,2,\cdots,N, \tag{25}$$

其中

$$A_{nm}^{(1)}=\int_{-h}^{h}[(q_1)_n^{(\alpha)}(p_1)_m^{(\beta)}+(q_2)_n^{(\alpha)}(p_2)_m^{(\beta)}+(q_3)_n^{(\alpha)}(p_3)_m^{(\beta)}+(q_4)_n^{(\alpha)}(p_4)_m^{(\beta)}]\mathrm{d}y,\qquad x=-l,$$

$$B_{nm}^{(1)}=\int_{-h}^{h}[(q_1)_n^{(\beta)}(p_1)_m^{(\beta)}+(q_2)_n^{(\beta)}(p_2)_m^{(\beta)}+(q_3)_n^{(\beta)}(p_3)_m^{(\beta)}+(q_4)_n^{(\beta)}(p_4)_m^{(\beta)}]\mathrm{d}y,\qquad x=-l,$$

$$A_{nm}^{(2)}=\int_{-h}^{h}[(p_1)_n^{(\alpha)}(q_1)_m^{(\alpha)}+(p_2)_n^{(\alpha)}(q_2)_m^{(\alpha)}+(p_3)_n^{(\alpha)}(q_3)_m^{(\alpha)}+(p_4)_n^{(\alpha)}(q_4)_m^{(\alpha)}-\eta_{34}(\partial_y q_3)_n^{(\alpha)}(q_4)_m^{(\alpha)}]\mathrm{d}y,\qquad x=l,$$

$$B_{nm}^{(2)}=\int_{-h}^{h}[(p_1)_n^{(\beta)}(q_1)_m^{(\alpha)}+(p_2)_n^{(\beta)}(q_2)_m^{(\alpha)}+(p_3)_n^{(\beta)}(q_3)_m^{(\alpha)}+(p_4)_n^{(\beta)}(q_4)_m^{(\alpha)}-\eta_{34}(\partial_y q_3)_n^{(\beta)}(q_4)_m^{(\alpha)}]\mathrm{d}y,\qquad x=l,$$

$$C_m^{(2)} = \int_{-h}^{h} [(q_1)_m^{(\alpha)} \tilde{\sigma}_x + (q_2)_m^{(\alpha)} \tilde{\sigma}_{xy} + (q_3)_m^{(\alpha)} \tilde{H}_x - \eta_{36} (q_4)_m^{(\alpha)} \tilde{H}_{xy} - \eta_{35} (q_4)_m^{(\alpha)} \tilde{\sigma}_{xy}] \mathrm{d}y, \qquad x = l.$$

5 数值结果及分析

为了方便数值计算,以 h 为特征长度,$(\lambda + 2\mu) \times 10^{-6}$ 为特征应力进行无量纲化.选取材料参数[29] $\lambda = 57.41$ GPa, $\mu = 88.45$ GPa, $R = -1.1$ GPa, $K_1 = 122$ GPa, $K_2 = 24$ GPa,取无量纲长度 $l/h = 4$.仍考虑平面矩形区域,左端位移条件和右端应力条件(24).首先需要验证和分析辛数值方法的收敛性.注意到,式(23)中前 6 项为零本征值本征解,之后的级数项为非零本征值本征解.因此级数收敛性取决于非零本征值本征解,即级数至少大于 6 项.为验证收敛性的合理性,以边界无量纲应力 $\tilde{\sigma}_x = 0.01(1 - Y^2)$,且其余载荷为 0 为例.由于非零本征值本征解主要反应边界效应,所以取靠近右端边界处位置作为观测点,如 $X = 3.9$ 和 $Y = 0.6$,并考察声子场正应力 σ_x 的收敛性及收敛数值.图 1 给出了该应力随级数项数 N 增大的规律.从该图中可以看出,$N > 21$ 以后基本稳定在 $\sigma_x = 6.6 \times 10^{-3}$,该数值与二维准晶有限元方法[30]的计算结果一致,验证了该数值方法的收敛性.同时也说明级数项数 N 与计算的精度密切相关.应该指出,级数的前 N 项包含了 $2N$ 个本征解.以下的计算不妨均取 $N = 30$.

为了验证辛方法的合理性和正确性,与二维准晶有限元方法[30]的计算结果对比.仍考虑左端固定,右端只有声子场载荷问题,即 $\tilde{\sigma}_x = 1$,其余荷载为 0.图 2 给出了 $y = 0$ 处的声子场位移 U_x.图中结果表明,本文数值结果与有限元计算结果相吻合,说明辛方法在求解二维准晶平面问题上是有效和可靠的.

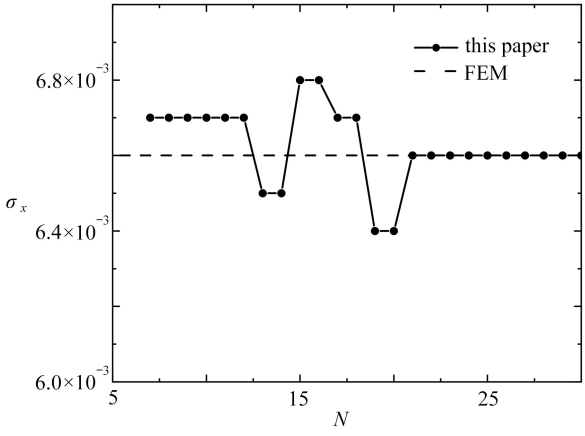


图 1 收敛性分析

Fig. 1 The convergence study

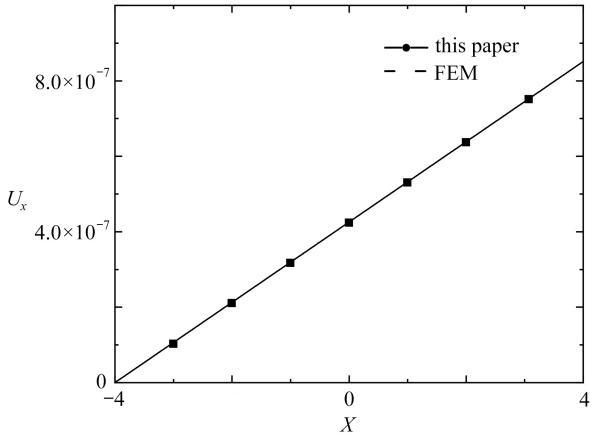


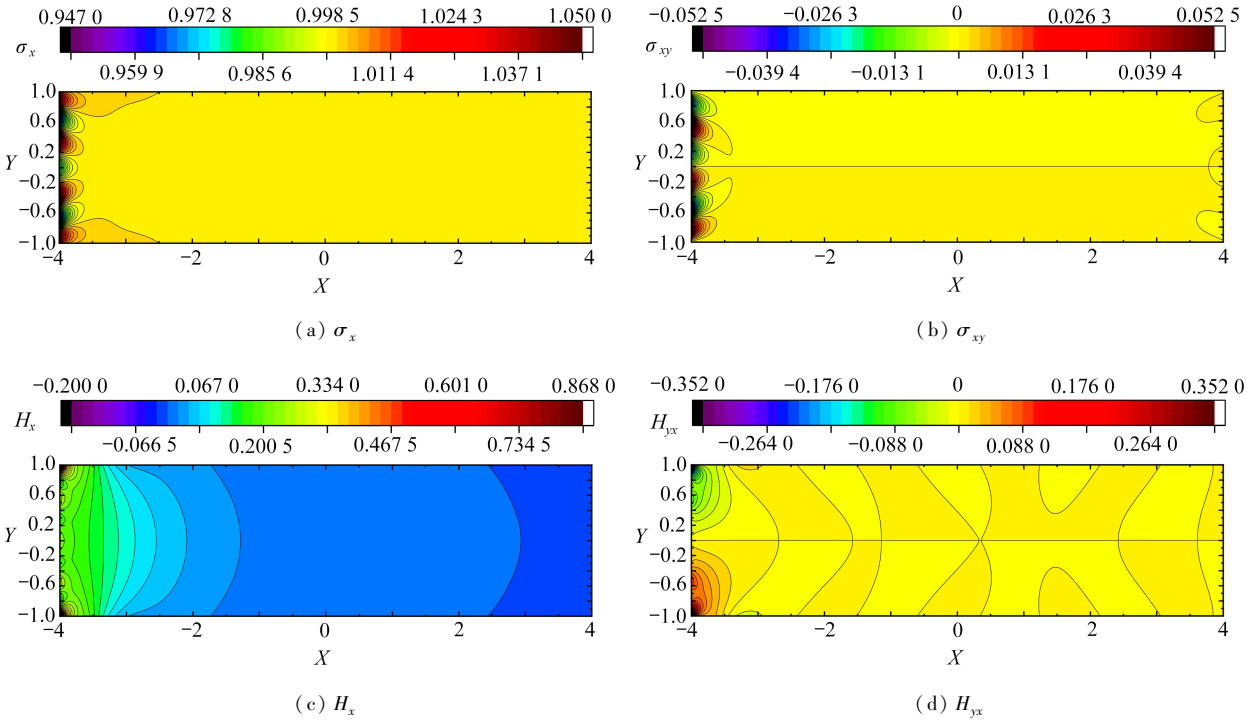
图 2 声子场位移对比

Fig. 2 Comparison of phonon field displacements

下面讨论不同载荷条件对位移和应力分布的影响.先设仅有声子场正应力 $\tilde{\sigma}_x$ 的条件.对于弹性问题,$\tilde{\sigma}_x$ 的大小与结果为线性比例关系,因此为说明问题,不妨取 $\tilde{\sigma}_x = 1$,其余应力 $\tilde{\sigma}_{xy} = 0, \tilde{H}_x = 0, \tilde{H}_{xy} = 0$.图 3 给出了应力(对偶变量)分布特点.从图中可以看出,声子场和相位子场无量纲正应力(图 3(a)和图 3(c))关于 x 轴对称且在右端($X = 4$)应力为常数.而在左端($X = -4$)应力呈现非均匀性,这是由于位移的固定限制了变形的缘故.由图 3(b)和图 3(d)也可以看出,声子场和相位子场切应力呈现反对称性.

注意到,图 3(b)、3(c)和 3(d),右端的应力非恒为零,此现象是由于方程(20)仅取 $N = 30$ 项.从理论上讲,无穷多项才是精确解.还注意到,左固定端($X = -4$)角点($Y = \pm 1$)附近出现应力集中现象,该应力集中数值是由项数累计而成.随项数增加,应力的奇异性更加明显.

考虑边界作用声子场剪应力 $\tilde{\sigma}_{xy}$ 对位移的影响,即在右端边界条件 $\tilde{\sigma}_x = 0, \tilde{\sigma}_{xy} = 1, \tilde{H}_x = 0, \tilde{H}_{xy} = 0$ 下位移的分布(图 4).由图 4 的 4 个分图可以看出,4 个位移分布呈现不同的特点.由于右端剪力载荷的作用,x 方向的声子场位移(图 4(a))和相位子场位移(图 4(c))表现出关于 x 轴反对称的特点,而 y 方向的位移(图 4(b)和 4(d))为对称的特点.这种现象是由位移和应力的对称性和反对称性关系决定的.图 4(a)和图 4(c)右端位移的反对称性是由整体变形产生的.

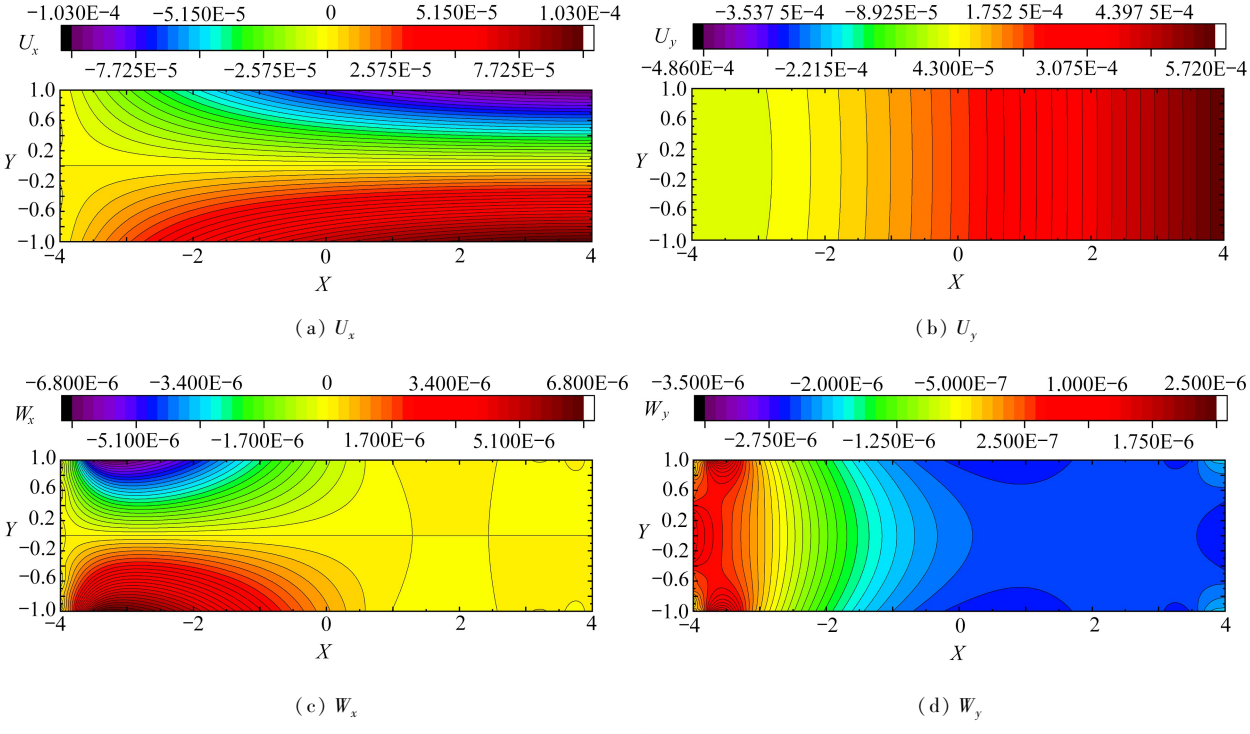


(a) σ_x　　(b) σ_{xy}

(c) H_x　　(d) H_{yx}

图 3　广义应力分布

Fig. 3　Generalized stress distributions



(a) U_x　　(b) U_y

(c) W_x　　(d) W_y

图 4　广义位移分布

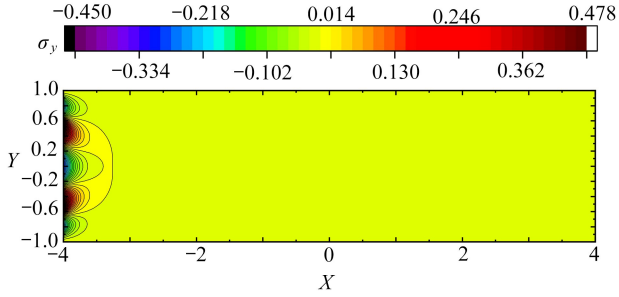
Fig. 4　Generalized displacement distributions

注　为了解释图中的颜色,读者可以参考本文的电子网页版本,后同.

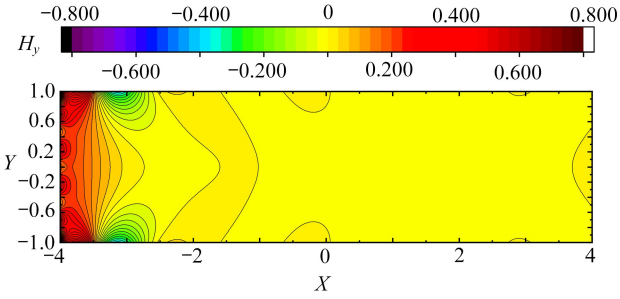
根据本构关系(2),除了对偶变量外,其他的应力可以由原变量和对偶变量表示为 $\sigma_y = \eta_2\sigma_x - \eta_5 H_x + \eta_{14}\partial_y u_y - \eta_{15}\partial_y w_y$, $H_y = \eta_3\sigma_x + \eta_6 H_x - \eta_{15}\partial_y u_y + \eta_{17}\partial_y w_y$ 和 $H_{xy} = -\eta_4\sigma_{xy} - \eta_7 H_{yx} + \eta_{16}\partial_y w_x$.为了显示这 3 个广义应力的分布特点,设右端边界条件 $\tilde{\sigma}_x = 0$, $\tilde{\sigma}_{xy} = 0$, $\tilde{H}_x = 1$, $\tilde{H}_{xy} = 0$.图 5 给出了这 3 个应力的分布图像.数

值结果表明,由于右端 $\tilde{H}_x = 1$ 的边界条件,应力 σ_y 和 H_y 呈现 y 轴的对称性,而应力 H_{xy} 表现为反对称性.

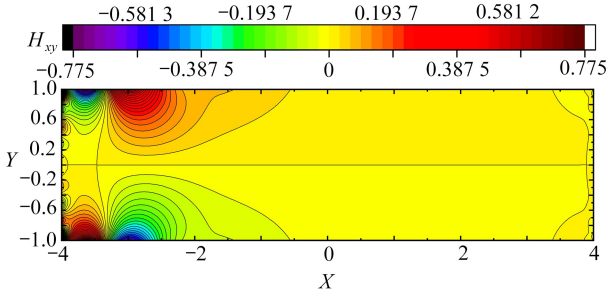
为了分析零本征值本征解和非零本征值本征解所起到的作用以及边界相位子场切应力 H_{xy} 对位移和应力的影响,给出如下算例.算例中不妨取右端边界条件为 $\tilde{\sigma}_x = 0, \tilde{\sigma}_{xy} = 0, \tilde{H}_x = 0, \tilde{H}_{xy} = 1$.



(a) σ_y



(b) H_y

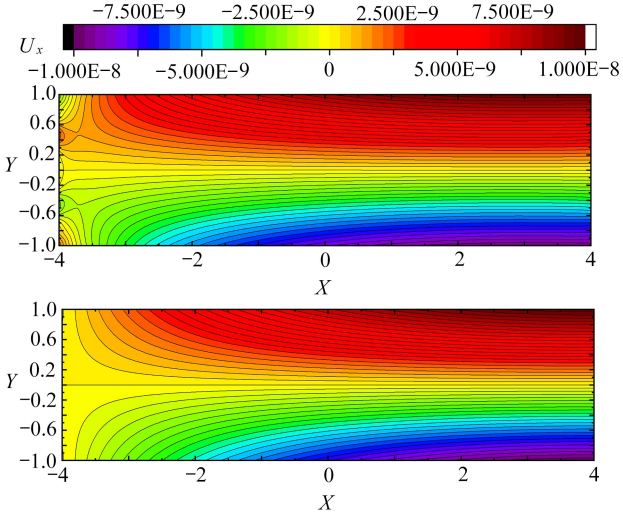


(c) H_{xy}

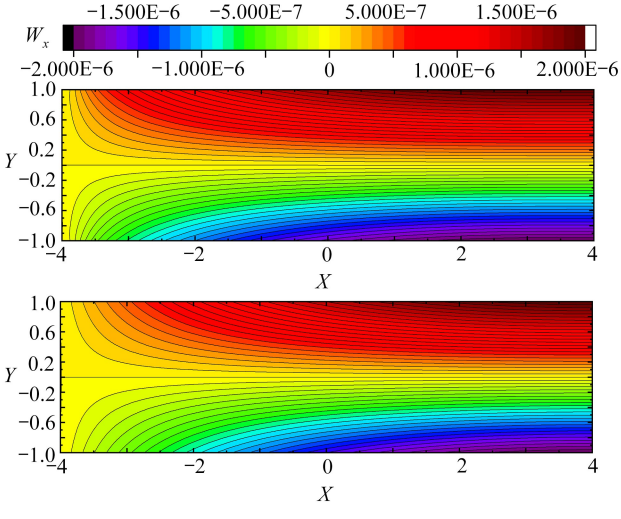
图 5 相关广义应力分布

Fig. 5 Correlative generalized stress distributions

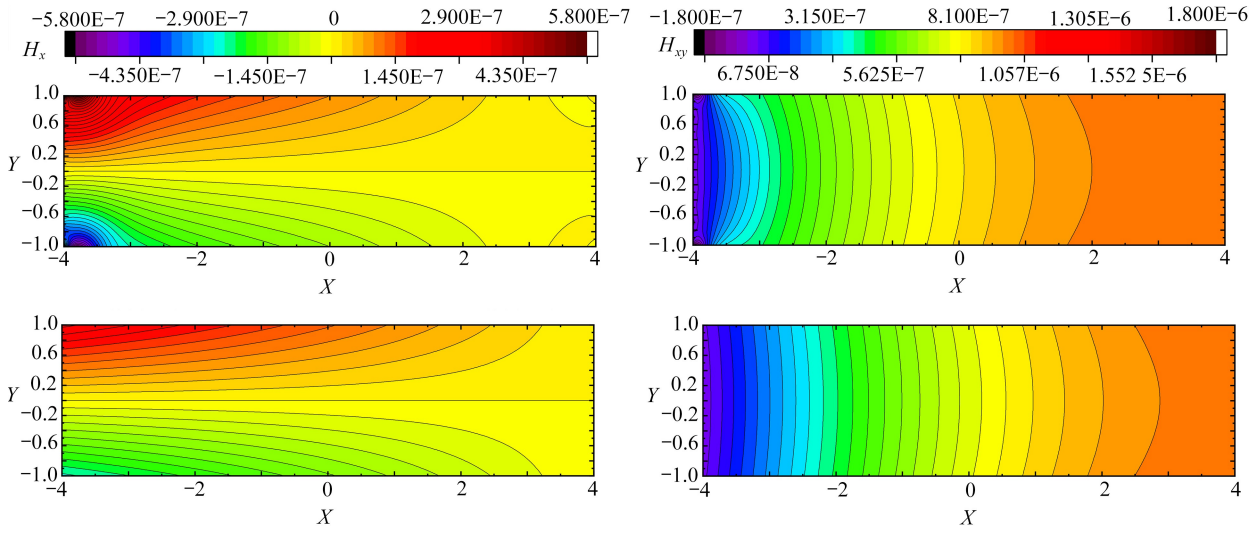
图 6 中每一个组合图的上图为完整解(零本征值本征解和非零本征值本征解),下图为仅由零本征值本征解描述的解.零本征值本征解刻画在合力情况下问题的解,该类情况属于 Saint-Venant 问题;而非零本征值本征解展示等效力系的偏差,或者说边界效应的影响,即 Saint-Venant 原理所覆盖的解.因此准确表述问题的解需要零本征值本征解和非零本征值本征解共同表示.由位移和应力对比图可以看出,在中部区域位移分布或应力分布几乎相同,可见零本征值本征解起主导作用.而在端部区域,位移分布或应力分布有一定的差别.这一点也恰好说明了零本征值本征解和非零本征值本征解的作用.应该指出,若随着区域长度的增加,区域内主要部分的位移和应力由零本征值本征解主导.



(a) U_x



(b) W_x



(c) H_x　　(d) H_{xy}

图 6　零本征值本征解和非零本征值本征解的作用

Fig. 6　The effects of zero-eigenvalue eigensolutions and non zero-eigenvalue eigensolutions

6　结　　论

在准晶问题中,声子场和相位子场的 4 个位移和 7 个应力相互关联,而在 Hamilton 体系下 4 个位移(原变量)仅需对应 4 个应力(对偶变量).除了对偶变量外,其他的应力可以由原变量和对偶变量表示.在 Hamilton 体系下,问题归结为零本征值本征解和非零本征值本征解问题.零本征值本征解描述了在合力情况下问题的解,属于 Saint-Venant 问题;而非零本征值本征解刻画了边界效应的影响,属于 Saint-Venant 原理所覆盖的解.随着区域长度的增加,区域内主要部分的位移和应力由零本征值本征解起主导作用.数值结果表明,结果的精度与所取本征解的项数有关,无穷多项才是精确解.由于固定位移边界条件限制了局部变形,因而会产生局部的应力集中现象.虽然辛本征解中不含有奇异项,此应力集中数值是靠所取辛本征解项数累计而得.求解过程和数值结果说明,Hamilton 体系方法是一种直接的解析方法,并可拓广到其他种类的准晶问题和空间问题.

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参考文献(References):

[1]　范天佑. 准晶数学弹性理论和某些有关研究的进展(上)[J]. 力学进展, 2012, **42**(5): 501-521.(FAN Tianyou. Development on mathematical theory of elasticity of quasicrystals and some relevant topics (I)[J]. *Advances in Mechanics*, 2012, **42**(5): 501-521.(in Chinese))

[2]　范天佑. 准晶数学弹性理论和某些有关研究的进展(下)[J]. 力学进展, 2012, **42**(6): 675-691.(FAN Tianyou. Development on mathematical theory of elasticity of quasicrystals and some relevant topics (II)[J]. *Advances in Mechanics*, 2012, **42**(6): 675-691.(in Chinese))

[3]　DUBOIS J M. New prospects from potential applications of quasicrystalline materials[J]. *Materials Science and Engineering: A*, 2000, **294/296**: 4-9.

[4]　AMINI M, RAHIMIPOUR M R, TAYEBIFARD S A, et al. Towards physical and mechanical properties of the novel Al-Cr-Ni-Fe decagonal quasicrystal and crystalline approximants[J]. *Advanced Powder Technology*, 2022, **33**(2): 103383.

[5] TAKAGIWA Y, MAEDA R, OHHASHI S, et al. Reduction of thermal conductivity for icosahedral Al-Cu-Fe quasicrystal through heavy element substitution[J]. *Materials*, 2021, **14**(18): 5238.

[6] STROUD R M, VIANO A M, GIBBONS P C, et al. Stable Ti-based quasicrystal offers prospect for improved hydrogen storage[J]. *Applied Physics Letters*, 1996, **69**(20): 2998-3000.

[7] 康国政, 陈义甫, 黄伟洋. 介电高弹体的力-电耦合循环变形和疲劳失效行为研究[J]. 力学进展, 2023, **53**(3): 592-625.(KANG Guozheng, CHEN Yifu, HUANG Weiyang. Review on electro-mechanically coupled cyclic deformation and fatigue failure behavior of dielectric elastomers[J]. *Advances in Mechanics*, 2023, **53**(3): 592-625.(in Chinese))

[8] JARIĆ M V, NELSON D R. Diffuse scattering from quasicrystals[J]. *Physical Review B*, 1988. DOI:10.1103/PhysRevB.37.4458.

[9] FAN Tianyou. *Mathematical Theory of Elasticity of Quasicrystals and Its Applications*[M]. Berlin: Springer, 2011.

[10] DING D H, YANG W G, HU C Z, et al. Generalized elasticity theory of quasicrystals[J]. *Physical Review B: Covering Condensed Matter and Materials Physics*, 1993, **48**(10): 7003-7010.

[11] FAN T Y, GUO L H. The final governing equation and fundamental solution of plane elasticity of icosahedral quasicrystals[J]. *Physics Letters A*, 2005, **341**(1/4): 235-239.

[12] GAO Y, SHANG L G. Governing equations and general solutions of plane elasticity of two-dimensional decagonal quasicrystals[J]. *International Journal of Modern Physics B*, 2011, **25**(20): 2769-2778.

[13] ZHANG Liangliang, YANG Lianzhi, YU Lianying, et al. General solutions of thermoelastic plane problems of two-dimensional quasicrystals[J]. *Transactions of Nanjing University of Aeronautics and Astronautics*, 2014, **31**(2): 142-146.

[14] ZHAO X F, LI X, DING S H. Two kinds of contact problems in three-dimensional icosahedral quasicrystals [J]. *Applied Mathematics and Mechanics* (*English Edition*), 2015, **36**(12): 1569-1580.

[15] 李光芳, 刘昉昉, 于静, 等. 立方准晶压电材料的半空间问题[J]. 应用数学和力学, 2023, **44**(7): 825-833.(LI Guangfang, LIU Fangfang, YU Jing, et al. The half space problem of cubic quasicrystal piezoelectric materials [J]. *Applied Mathematics and Mechanics*, 2023, **44**(7): 825-833.(in Chinese))

[16] 杨震霆, 王雅静, 聂雪阳, 等. 含切口的压电准晶组合结构界面断裂分析的辛-等几何耦合方法[J]. 应用数学和力学, 2024, **45**(2): 144-154.(YANG Zhenting, WANG Yajing, NIE Xueyang, et al. Symplectic isogeometric analysis coupling method for interfacial fracture of piezoelectric quasicrystal composites with notches[J]. *Applied Mathematics and Mechanics*, 2024, **45**(2): 144-154.(in Chinese))

[17] FENG X, ZHANG L L, LI Y, et al. On the propagation of plane waves in cubic quasicrystal plates with surface effects[J]. *Physics Letters A*, 2023, **473**: 128807.

[18] 原庆丹, 郭俊宏. 一维纳米准晶层合梁的非局部振动、屈曲与弯曲研究[J]. 应用数学和力学, 2024, **45**(2): 208-219.(YUAN Qingdan, GUO Junhong. Nonlocal vibration, buckling and bending of 1D layered quasicrystal nanobeams[J]. *Applied Mathematics and Mechanics*, 2024, **45**(2): 208-219.(in Chinese))

[19] 范俊杰, 李联和, 阿拉坦仓. 对边简支十次对称二维准晶板弯曲问题的辛分析[J]. 应用数学和力学, 2023, **44**(7): 834-846.(FAN Junjie, LI Lianhe, ALATANCANG. Symplectic analysis on the bending problem of decagonal symmetric 2D quasicrystal plates with 2 opposite edges simply supported[J]. *Applied Mathematics and Mechanics*, 2023, **44**(7): 834-846.(in Chinese))

[20] 王会苹, 王桂霞, 陈德财. 含椭圆孔有限大二十面体准晶板平面弹性问题的边界元分析[J]. 应用数学和力学, 2024, **45**(4): 400-415.(WANG Huiping, WANG Guixia, CHEN Decai. Boundary element analysis for the plane elasticity problems of finite icosahedral quasicrystal plates containing elliptical holes[J]. *Applied Mathematics and Mechanics*, 2024, **45**(4): 400-415.(in Chinese))

[21] ZHU S B, TONG Z Z, LI Y Q, et al. Post-buckling of two-dimensional decagonal piezoelectric quasicrystal cylindrical shells under compression[J]. *International Journal of Mechanical Sciences*, 2022, **235**: 107720.

[22] ZHONG W X. *Duality System in Applied Mechanics and Optimal Control*[M]. Boston：Kluwer Academic Publishers, 2004.

[23] WANG H, LI L H, HUANG J J, et al. Symplectic approach for the plane elasticity problem of quasicrystals with point group 10 mm[J]. *Applied Mathematical Modelling*, 2015, **39**(12)：3306-3316.

[24] QIAO Y F, HOU G L, CHEN A. Symplectic approach for plane elasticity problems of two-dimensional octagonal quasicrystals[J]. *Applied Mathematics and Computation*, 2021, **400**：126043.

[25] SUN Z Q, HOU G L, QIAO Y F, et al. Hamiltonian system for the inhomogeneous plane elasticity of dodecagonal quasicrystal plates and its analytical solutions[J]. *Chinese Physics B*, 2024, **33**(1)：016107.

[26] LI G F, LI L H. An analysis method of symplectic dual system for decagonal quasicrystal plane elasticity and application[J]. *Crystals*, 2022, **12**(5)：636.

[27] 郭丽辉, 范天佑. 准晶弹性理论边值问题的可解性[J]. 应用数学和力学, 2007, **28**(8)：949-957.(GUO Lihui, FAN Tianyou. Solvability on boundary-value problems of elasyicity of three-dimensional quasicrystals[J]. *Applied Mathematics and Mechanics*, 2007, **28**(8)：949-957.(in Chinese))

[28] CAO H B, SHI Y Q, LI W. Analytic solutions to two-dimensional decagonal quasicrystals with defects using complex potential theory[J]. *Crystals*, 2019, **9**(4)：209.

[29] LI W, FAN T Y. Plastic analysis of the crack problem in two-dimensional decagonal Al-Ni-Co quasicrystalline materials of point group[J]. *Chinese Physics B*, 2011, **20**(3)：036101.

[30] LI T, YANG Z T, XU C H, et al. A phase field approach to two-dimensional quasicrystals with mixed mode cracks[J]. *Materials*, 2023, **16**(10)：3628.